Overview of Lecture 6 (Lecture 5 in Reader)

- Annuities and Perpetuities
- Car Loan and Retirement Planning

Perpetuities

- A **perpetuity** is a financial instrument that pays C dollars per period forever.
- If the interest rate is constant and the first payment from the perpetuity arrives in period 1, then the PV of the perpetuity is:

$$PV = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

- E.g. C = $100, and r = 20%.
  - PV = 100/0.2 = 500.

Growing Perpetuities

- A **growing perpetuity** pays $C(1+g)^{t-1}$ per period.
- Example: if C=100 and g=10%, the payments are:
  - $C_1 = 100$
  - $C_2 = 100(1.1) = 110$
  - $C_3 = 100(1.1)^2 = 121$, etc.
- Its PV is (see BM p. 41):

$$PV = \sum_{t=1}^{\infty} \frac{C(1+g)^{t-1}}{(1+r)^t} = \frac{C}{r-g}$$

- In the example, if r=20%,

$$PV = \frac{C}{r-g} = \frac{\$100}{0.2-0.1} = \$1,000.$$

Annuity

- An **annuity** is a financial instrument that pays C dollars for T periods.
  - It has the following PV formula (see BM p. 42):

$$PV = \sum_{t=1}^{T} \frac{C}{(1+r)^t} = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

- A **growing annuity** pays $C(1+g)^{t-1}$ per period starting in period 1 for T periods. Its PV is

$$PV = \sum_{t=1}^{T} C\frac{(1+g)^{t-1}}{(1+r)^t} = \frac{C}{(r-g)}\left[1 - \frac{(1+g)^T}{(1+r)^T}\right].$$

  - See reader, p. 37, for a "trick" for doing this on a calculator.

Annuity example – Calculating monthly payments on car loan

- Loan amount = \$8,239.05
- 4 years to maturity
- 48 equal monthly payments
  - The loan is an annuity with 48 monthly payments, starting one month from today.
- Quoted (APR) interest rate of 14.2%.
  - We must use the monthly interest rate:
  - $r_{monthly}$ = APR/12 = 14.2%/12 = 1.183% per month
- How much do you pay each month?

Annuity example – Calculating monthly payments on car loan

- The amount borrowed (8,239.05) equals the PV of all future payments, calculated using the loan rate.
- From formula:

$$8{,}239.05 = C/1.01183 + C/(1.01183)^2 + \ldots + C/(1.01183)^{48}$$
$$= C/.01183\ [1 - 1/(1.01183)^{48}]$$
$$= C \times 36.46$$

- $36.46 is sometimes called an **Annuity Factor**.
- So C = 8239.05 / 36.46 = $225.97.
- On a calculator:

  n = 48, i = 1.183333%, PV = $8,239.05.

  [PMT] → $225.97.

Loans:
Principal and Interest

- Of the first 3 month's payments, how much is interest and how much is principal? What is the remaining principal?

Date	Payment (PMT)	Interest (I)	Principal (P)	Balance (B)
		$.01183 * B_{t-1}$	PMT - I	B_{t-1} - P
0				8,239.05
1	225.97	97.50	128.48	8,110.57
2	225.97	95.98	130.00	7,980.58
3	225.97	94.44	131.53	7,849.04
. . .				
46	225.97	7.84	218.14	444.05
47	225.97	5.25	220.72	223.33
48	225.97	2.64	223.33	0.00

Loans: Computing Remaining Principal

- We have seen B_3= $7,849.04 from the spreadsheet.
- OR… The remaining balance on a loan can be calculated as the PV of <u>remaining</u> payments, using the loan rate.

$$B_3 = 225.97/1.01183 + 225.97/(1.01183)^2 + \ldots + 225.97/(1.01183)^{45}$$
$$= 225.97/.01183\ [1 - 1/(1.01183)^{45}]$$
$$= \$7{,}849.04.$$

- Or, on a calculator:

  n = 45, i = 1.183333%, PMT = 225.97(167).

  [PV] → $7,849.04.

Growing Annuity Beginning at a Future Date

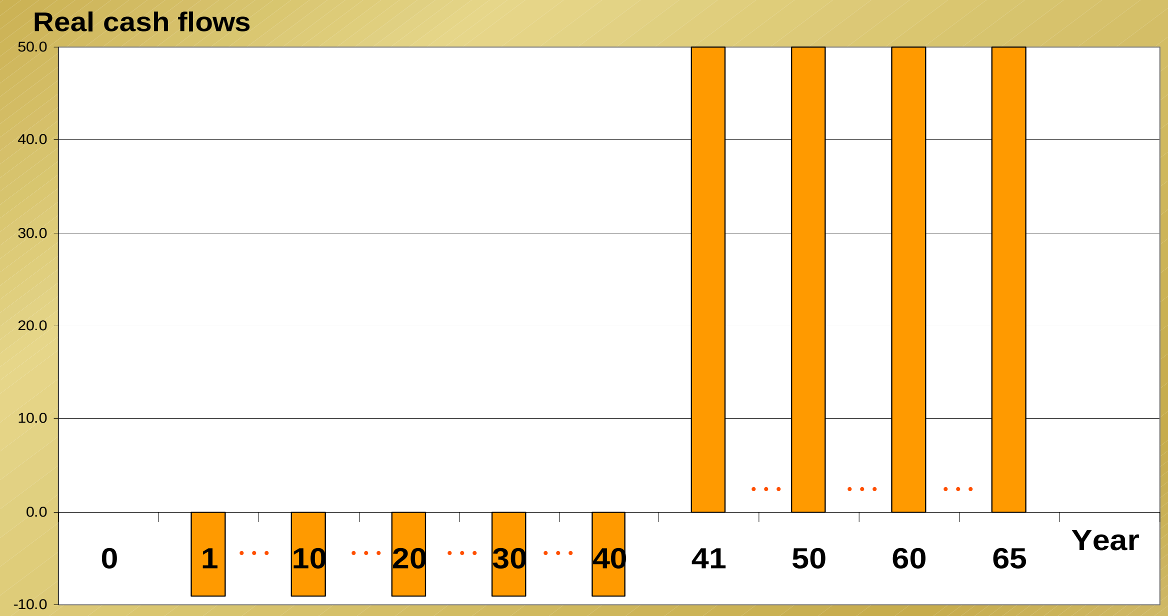
- If a T period annuity has its first payment (C) arrive in period τ, then its value equals

$$PV = \sum_{t=\tau}^{T+\tau-1} C\frac{(1+g)^{t-\tau}}{(1+r)^{t}} = \frac{C}{(r-g)(1+r)^{(\tau-1)}}\left[1-\frac{(1+g)^{T}}{(1+r)^{T}}\right].$$

- I.e. first compute PV as of period τ-1, then discount back from period τ-1 to today.
- E.g. you will receive $10 in year 3, which will grow at 15% a year for a total of 5 payments. r = 10%.
  - PV as of period <u>2</u> = 10 / (.1 - .15) [1 – $\{1.15/1.10\}^5$] = $49.78
  - PV as of period <u>0</u> = 49.78 / 1.1^2 = $41.14

The Retirement Problem

- Thad Johnson is 25 years old. He will retire at age 65, die at age 90.
- Date 0 is now; work till year 40; die at year 65.
- Goal: Income of real $50,000/year for retirement years.
- Plan: Save constant $/year (real) to fund retirement goal.



Retirement Problem: Returns and Inflation

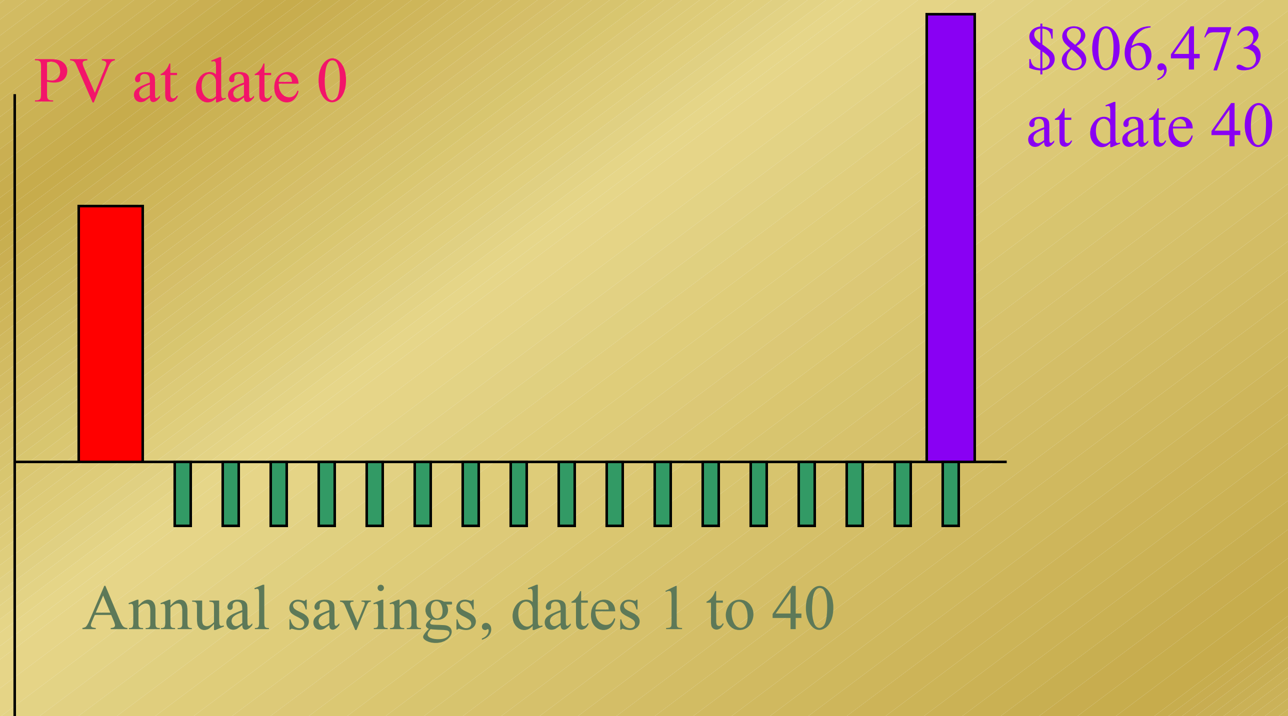
- Investment return =12%; inflation rate = 8%.
- Problem is given in real terms, so use real rates: r_{real} = real interest rate, r = nominal interest rate, and i = inflation rate.
- From our formula: $r_{real} = (r - i)/(1+i)$

  $r_{real} = (.12-.08)/1.08 = .03704 = 3.704\%$ annually.

Retirement Solution (Part A): How Much Needed by Date 40?

- Consider problem from his perspective at age 65.
  - Amount in bank must be enough to pay annual annuity of \$50,000 (real) per year for 25 years.
  - I.e. he needs to have saved the PV of the annuity as of age 65.
- $PV_{65} = 50{,}000/1.037 + \ldots + 50{,}000/(1.037)^{25}$
  $= 50{,}000 / .037\ [1 - (1/1.037)^{25}]$
  $= \$806{,}473.$
- Calculator:
  - N=25, i =3.7%, PMT = 50,000, FV = 0;
  - Compute PV_{65} = 806,473.

Retirement Solution (Part B): How much annual saving?



- PV of annual savings must equal PV of amount in bank at date 40.

Retirement Solution (Part B): How much Annual Saving?

- First, compute PV of retirement fund of 806,473.
  - PV = 806,473 / 1.037^{40} = \$188,556.
- Next, compute annual payment:
  - $188{,}556 = C / 1.037 + C / 1.037^2 + \ldots + C / 1.037^{40}$
    $= C / .037\,[1 - (1/1.037)^{40}] = C \times 20.71$
  - So C = 188,556 / 20.71 = \$9,105.
- Or, on calculator:
  - n = 40,   i = 3.7%,   FV = 0,   PV = 188,556
  - [PMT] → C = 9,105.