Lecture 11

- Staticstics review and portfolio theory

- Readings:
  - Reader, lecture 13
  - BM Chapter 7

Where are we?

- We know, in principle, how to make investment decisions:
  - PV/NPV
  - Discount cash flows using an appropriate discount rate
- Where does the discount rate come from?
- We need the **Capital Asset Pricing Model (CAPM)**
  - Gives quantitative tradeoff between **risk** and **expected return**.

What we need to know

- How to calculate the **expected return** on a stock.
- How to calculate the **variance** of stock returns.
- How to apply the above to a portfolio of stocks:
  - How to calculate the **expected return on a portfolio**.
  - How to calculate the **covariance** between the return on two stocks.
  - How to calculate the **variance of a portfolio**.

A Single Stock: Notation and Definitions

- Notation:
  - $\tilde{r}_i$ is the (random) return on stock i.
  - r_{ix} is one possible return for stock i, which occurs with probability p_x
  - $\bar{r}_i$ is the expected return on stock i.

$$\text{Expected Return} = E(\tilde{r}_i) = \bar{r}_i = \sum_x p_x r_{ix}$$

$$\text{Variance} = \text{Var}(\tilde{r}_i) = \sigma_i^2 = E(\tilde{r}_i - \bar{r}_i)^2 = \sum_x p_x (r_{ix} - \bar{r}_i)^2$$

$$\text{Standard Deviation} = \sigma_i = \sqrt{\text{Variance}}$$

A Single Stock: Expected Return

$$\text{Expected Return} = E(\tilde{r}_i) = r_i = \sum_{x} p_x r_{ix}$$

r_{ix}	p_X	$p_x r_{ix}$
-5	0.25	-1.25
0	0.25	0
5	0.25	1.25
12	0.25	3
	$r_i=$	3

A Single Stock: Variance

$$\text{Stock Variance} = \text{Var}(\tilde{r}_i) = E(\tilde{r}_i - \bar{r}_i)^2 = \sum_x p_x (r_{ix} - \bar{r}_i)^2$$

r_{ix}	P_x	$(r_{ix} - r_i)^2$	$p_x(r_{ix} - r_i)^2$
-5	0.25	64	16.00
0	0.25	9	2.25
5	0.25	4	1.00
12	0.25	81	20.25
		Var =	39.50

Portfolio Weights

- Take a portfolio worth \$200 that contains
  - \$40 worth of stock A,
  - \$60 worth of stock B,
  - \$100 worth of stock C.
- Then the portfolio has investment weights of:
  - $w_A = 40/200 = .20$ in stock A
  - $w_B = 60/200 = .30$ in stock B
  - $w_C = 100/200 = .50$ in stock C
- The weights on any portfolio must sum to 1

Portfolios: Expected Return

- Formula for the (expected) return on a portfolio:

$$\tilde{r}_p = w_i \tilde{r}_i + w_j \tilde{r}_j \text{ (actual returns)}$$

$$r_p = w_i r_i + w_j r_j \text{ (expected returns)}$$

- Notation:
  - the expected return on stock i is r_i,
  - the expected return on stock j is r_j,
  - the weight for stock i is w_i,
  - the weight for stock j is w_j.
- Example: $r_i = 3$, $r_j = 5$, $w_i = .4$, and $w_j = .6$. Then:

$$r_p = (.4)(3) + (.6)(5) = 4.2.$$

CD and NBF Sample Problem: Finding Portfolio Weights

- Invest in Corporate Disasters (CD) at 4%.
- Or, invest in Nevada beach front property (NBF): Investment will have an expected return of 10%.

- You want a portfolio with expected return 6%.
- What portfolio fraction should you invest in the Nevada property?

Finding Portfolio Weights: Solution

- Use formula for expected return on a portfolio
- $6 = w_{CD}(4) + w_{NBF}(10)$
- Since $w_{CD} + w_{NBF} = 1$, $w_{CD} = 1 - w_{NBF}$.
- So we can write $6 = (1 - w_{NBF})(4) + w_{NBF}(10)$
- We find $w_{NBF} = 1/3$

Covariance: The relationship between two returns

- A zero **covariance** implies no relationship.
- A positive covariance implies that when stock i has an exceptionally high return, so (usually) does stock j.
- A negative covariance implies that when the return on stock i is unusually high, the return on stock j tends to be unusually low.

$$\mathrm{Cov}\left(\tilde{r}_i, \tilde{r}_j\right) = E\left(\tilde{r}_i - r_i\right)\left(\tilde{r}_j - r_j\right) = \sigma_{ij} = \sum_x \sum_y p_{xy}\left(r_{ix} - r_i\right)\left(r_{jy} - r_j\right).$$

Covariance vs. Correlation

- People often talk about the **correlation** between stock i and j (ρ_{ij}) instead of their **covariance** (σ_{ij}).
- The two measures are very closely related.
- We can easily convert from one to the other:

$$\rho_{ij} = \frac{\mathrm{Cov}(\tilde{r}_i, \tilde{r}_j)}{\mathrm{SD}(\tilde{r}_i)\mathrm{SD}(\tilde{r}_j)} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}.$$

Example: Computing covariance between two stocks

- Probabilities for returns on each stock

Joint Probabilities for each pair of returns on stocks i and j		r_{jy} Returns on j		
		2	7	8
r_{ix} Returns on i	0	0.1	0	0
	6	0	0.2	0.1
	12	0	0.2	0.4

Example 1: Computation Matrix for Variance and Covariance

r_{ix}	r_{jy}	p_{xy}	$p_{xy}\ r_{ix}$	$p_{xy} r_{jy}$	$r_{ix} - r_i$	$r_{jy} - r_j$	$p_{xy}(r_{ix} - r_i)(r_{jy} - r_j)$
0	2	0.1	0.0	0.2	-9	-5	4.5
6	7	0.2	1.2	1.4	-3	0	0.0
12	7	0.2	2.4	1.4	3	0	0.0
6	8	0.1	0.6	0.8	-3	1	-0.3
12	8	0.4	4.8	3.2	3	1	1.2
			$r_i =$ 9.0	$r_j =$ 7.0			$Cov(\tilde{r}_i, \tilde{r}_j) =$ 5.4

Rules Governing Portfolio Covariance and Variance

- Rules for covariance:

$$\mathrm{cov}\left(w_i \tilde{r}_i, w_j \tilde{r}_j\right) = w_i w_j \mathrm{cov}\left(\tilde{r}_i, \tilde{r}_j\right)$$

$$\mathrm{cov}\left(\tilde{r}_i, \tilde{r}_j + \tilde{r}_k\right) = \mathrm{cov}\left(\tilde{r}_i, \tilde{r}_j\right) + \mathrm{cov}\left(\tilde{r}_i, \tilde{r}_k\right)$$

$$\mathrm{var}\left(\tilde{r}_i\right) = \mathrm{cov}\left(\tilde{r}_i, \tilde{r}_i\right)$$

- The variance of a portfolio with two stocks:

$$\mathrm{var}\left(w_i \tilde{r}_i + w_j \tilde{r}_j\right) = w_i^2 \ \mathrm{var}\left(\tilde{r}_i\right) + w_j^2 \mathrm{var}\left(\tilde{r}_j\right) + 2 w_i w_j \mathrm{cov}\left(\tilde{r}_i, \tilde{r}_j\right)$$

Example 1: Computing Portfolio Expected Return and Variance

- In previous example, we found $r_i = 9.0$, $r_j = 7.0$.
- If we form a portfolio with $w_i = .4$, $w_j = .6$, then

$$r_p = E\left(.4\tilde{r}_i + .6\tilde{r}_j\right) = .4(9) + .6(7) = 7.8.$$

- For homework, show that $\text{var}\left(\tilde{r}_i\right) = 16.2$, and $\text{var}\left(\tilde{r}_j\right) = 3.$
- We also know that $\text{cov}(\tilde{r}_i, \tilde{r}_j) = 5.4$, so

$$\begin{aligned}\text{var}(\tilde{r}_p) &= \text{var}\left(.4\tilde{r}_i + .6\tilde{r}_j\right)\\ &= .4^2(16.2) + .6^2(3) + 2(.4)(.6)(5.4)\\ &= 6.264.\end{aligned}$$

Multiple Stock Portfolios: Expected Return and Variance

- Expected return on portfolio, r_P:

$$r_p = w_1 r_1 + w_2 r_2 + ... + w_n r_n$$

Just plug in values and add terms together.

- Variance on portfolio, Var_P:

$$Var_p = Var\left(w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + ... + w_n \tilde{r}_n\right)$$

Use "box method" to compute portfolio variance.

$$\text{Notation}: \sigma_i^2 = \text{variance}_i;\ \sigma_i = \text{SD}_i;\ \sigma_{ij} = \text{covariance}_{ij}$$

Multiple Stock Portfolios: Computing Variance

- Add up all boxes in the weighted variance-covariance matrix:

	Stock				
		1	2	…	n
	1	$w_1^2\sigma_1^2$	$w_1 w_2 \sigma_{12}$	…	$w_1 w_n \sigma_{1n}$
	2	$w_1 w_2 \sigma_{12}$	$w_2^2\sigma_2^2$		$w_2 w_n \sigma_{2n}$
	.	.	.		.
Stock	n	$w_1 w_n \sigma_{1n}$	$w_2 w_n \sigma_{2n}$		$w_n^2\sigma_n^2$

Multiple Stock Portfolios: Example with 2 Stocks

$$\mathrm{Var}(w_1\tilde{r}_1 + w_2\tilde{r}_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$
$$= w_1^2\mathrm{Var}(\tilde{r}_1) + w_2^2\mathrm{Var}(\tilde{r}_2) + 2w_1w_2\mathrm{Cov}(\tilde{r}_1, \tilde{r}_2)$$

		STOCK	
		1	2
Stock	1	$w_1^2\sigma_1^2$	$w_1w_2\sigma_{12}$
	2	$w_1w_2\sigma_{12}$	$w_2^2\sigma_2^2$

Numerical Example 1

- What is the variance of a portfolio with:

$$w_1 = .2,\ w_2 = .8,\ \sigma_1^2 = 10,\ \sigma_2^2 = 20, \text{ and } \sigma_{12} = 5?$$

		STOCK	
		1	2
Stock	1	$(.2^2)10$	$(.2)(.8)5$
	2	$(.2)(.8)5$	$(.8^2)20$

$$\sigma_p^2 = (.2^2)10 + (.8^2)20 + 2(.2)(.8)5 = 14.8$$

Numerical Example 2

- You have a portfolio of 15 stocks:
  - The first 5 stocks have portfolio weights 0.1 (10%);
  - The remaining stocks have portfolio weight .05 (5%);
  - Each stock has a variance of 10,
  - All stocks have a covariance of 2 with each other.
- What is the variance of your portfolio?
  - To solve this problem, set up the matrix, and use the patterns in the matrix to simplify the box addition.

Numerical Example 2, Continued

	1	…	5	6	…	15
1	$(.1^2)(10)$		$(.1^2)(2)$	(.1)(.05)(2)		(.1)(.05)(2)
…						
5	$(.1^2)(2)$		$(.1^2)(10)$	(.1)(.05)(2)		(.1)(.05)(2)
6	(.1)(.05)(2)		(.1)(.05)(2)	$(.05^2)(10)$		$(.05^2)(2)$
…						
15	(.1)(.05)(2)		(.1)(.05)(2)	$(.05^2)(2)$		$(.05^2)(10)$

- Top left hand group of 25 boxes (stocks 1-5): The 5 diagonals are $(.1^2)(10)$. The remaining 20 entries are $(.1^2)(2)$. Total = $5\text{x}.1^2\text{x}10 + 20\text{x}.1^2\text{x}2 = .9$.

- Bottom right hand 100 boxes (stocks 6-15): The 10 diagonals = $(.05^2)(10)$. The other 90 entries are $.05^2\text{x}2$. Total = $10\text{x}.05^2\text{x}10 + 90\text{x}.05^2\text{x}2 = .7$.

- Top right and bottom left group for total of 100 boxes. Every box has the entry (.1)(.05)(2). Total = 100x.1x.05x2 = 1.

- Total portfolio variance = .9 + .7 +1 = 2.6.

Example: Eliminating “firm specific risk” via Diversification

- Consider a special portfolio, in which $w_i = 1/n$.

	STOCK				
Stock		1	2	…	n
	1	σ_1^2/n^2	σ_{12}/n^2	…	σ_{1n}/n^2
	2	σ_{12}/n^2	σ_2^2/n^2		σ_{2n}/n^2
	. .	. .	. .		. .
	n	σ_{1n}/n^2	σ_{2n}/n^2		σ_n^2/n^2

- For portfolio variance, add up the boxes.

Computing Variance of the Fully Diversified Portfolio

- Write the sum of the diagonal elements as

$$\sum_{i=1}^{n} \frac{\sigma_i^2}{n^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_i^2}{n} = \frac{1}{n}(\text{avg. variance}).$$

- Now add up off-diagonal elements. There are n x n boxes in total of which n are diagonals leaving n x n - n off diagonal boxes. Add up the off-diagonal boxes to get

$$\sum_{\text{off-diagonal boxes}} \frac{\sigma_{ij}}{n^2} = \frac{n^2 - n}{n^2} \sum_{\text{off-diagonal boxes}} \frac{\sigma_{ij}}{n^2 - n} = \left(1 - \frac{1}{n}\right)(\text{avg. cov.})$$

Variance of the Fully Diversified Portfolio: Conclusion

- Therefore the variance of the portfolio equals

$$\mathrm{Var}\left(\frac{1}{n}\tilde{r}_1 + \frac{1}{n}\tilde{r}_2 + \ldots + \frac{1}{n}\tilde{r}_n\right) = \frac{1}{n} \text{ x Avg.Var} + \left(1 - \frac{1}{n}\right) \text{x Avg.Cov.}$$

- So what is the variance of a "perfectly" diversified portfolio? This occurs as n goes to infinity. In that case:

$$\mathrm{Var}\left(\frac{1}{n}\tilde{r}_1 + \frac{1}{n}\tilde{r}_2 + \ldots + \frac{1}{n}\tilde{r}_n\right) = \text{Avg.Cov.}$$

Variance of a Fully Diversified Portfolio: Comment

- Note that **diversification** eliminates all risk except the average covariance of the stocks.
- Since people can do this for themselves, we'll see that
  - The market only rewards people for holding "market risk" (covariance between all stocks)
  - No reward for "firm specific risk" (variance of a single stock).